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Homotopy and K-theory seminar
Talk #5
October 3, 2018

Abstract

Even more basic category theory. The main sources for these notes are nLab, Rognes, Ch. 4, and Peter Johnstone's Part III lecture notes (Michaelmas 2015), Ch. 4.

Definition. An object X of $\mathscr{C}$ is *initial* if for each $Y \in \operatorname{ob}\mathscr{C}$, there is a unique morphism $f : X \to Y$. Moreover, we say that X is *terminal* if for each $Z \in \operatorname{ob}\mathscr{C}$, there is a unique morphism $g : Z \to X$. Either condition is called a *universal property* of X.

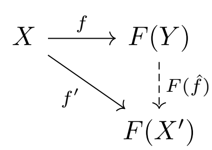
Definition. Any property P of $\mathscr{C}$ has a dual property P^{op} of $\mathscr{C}^{op}$ obtained by interchanging the source and target of any arrow as well as the order of any composition in the sentence expressing P. Then P is true of $\mathscr{C}$ iff P^{op} is true of $\mathscr{C}^{\mathrm{op}}$.

Lemma 1. Being initial and being terminal are dual properties.

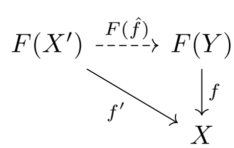
Lemma 2. Any two initial objects of $\mathscr{C}$ are canonically isomorphic. The same holds for any two terminal objects of $\mathscr{C}$.

Proof. Compose the two unique morphisms to get an isomorphism between the two initial objects. Apply duality to get the second claim. □

Remark 1. Think of a universal property as follows. Let $F : \mathscr{D} \to \mathscr{C}$ be a functor and $X \in \operatorname{ob}\mathscr{C}$. A *universal arrow from X to F* is an ordered pair (Y, f) with $Y \in \operatorname{ob}\mathscr{D}$ and $f : X \to F(Y)$ a morphism of $\mathscr{C}$ with the property that for any $X' \in \operatorname{ob}\mathscr{D}$ and morphism $f' : X \to F(X')$ of $\mathscr{C}$, there exists a unique morphism $\hat{f} : Y \to X'$ of $\mathscr{D}$ such that $F(\hat{f}) \circ f = f'$.



Dually, a *universal arrow from F to X* is an ordered pair (Y, f) with $Y \in \operatorname{ob}\mathscr{D}$ and $f : F(Y) \to X$ of $\mathscr{C}$ with the property that for any $X' \in \operatorname{ob}\mathscr{D}$ and morphism $f' : F(X') \to X$, there exists a unique morphism $\hat{f} : X' \to Y$ such that $f' = f \circ F(\hat{f})$.



Remark 2. To see why this notion of universality agrees with the original one, we first generalize the notion of an arrow category.

Definition. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor and $Y \in \operatorname{ob}\mathscr{D}$. The *slice* or *left fiber category*, denoted by (F/Y) or $(F \downarrow Y)$, has as objects pairs (X, f) where $f : F(X) \to Y$ and as morphisms from $f : F(X) \to Y$ to $f' : F(X') \to Y$ morphisms $g : X \to X'$ such that $f = f' \circ F(g)$.

Definition. The *coslice* or *right fiber category*, denoted by (Y/F) or $(Y \downarrow F)$, has as objects pairs (X, f) where $f : Y \to F(X)$ and as morphisms from $f : Y \to F(X)$ to $f' : Y \to F(X')$ morphisms $g : X \to X'$ such that $f' = F(g) \circ f$.

Remark 3. If $F^{op} : C^{op} \to D^{op}$ is opposite to the functor $F : \mathscr{C} \to \mathscr{D}$ and $Y \in \operatorname{ob}\mathscr{D}$, then $(Y/F)^{op} = F^{op}/Y$. Thus, the left and right fiber categories are dual in the sense that $P(Y, F)$ is true for any right fiber category Y/F iff $P^{op}(Y, F)$ is true for any left fiber category F/Y.

Proposition 1. Let $F : \mathscr{D} \to \mathscr{C}$ be a functor and $x \in \operatorname{ob} C$. Then $u : x \to Fr$ is a universal arrow from x to F iff it is initial object of the coslice $(x \downarrow F)$. Dually, $u' : Fr' \to x$ is a universal arrow from F to x iff it is a terminal object of the same category.

Proof. [[I messed this up during my talk. It should be correct as written now.]] Suppose that u is universal and $f : x \to Fy$ is another object of $(x \downarrow F)$. Then there is some unique $\hat{f} : r \to y$ such that $F(\hat{f}) \circ u = f$. Thus $F(\hat{f})$ is a unique morphism of the coslice.

Conversely, suppose that u is initial. Then for any object $f : x \to Fy$ of $(x \downarrow F)$, there is some unique arrow $Sg : Fr \to Fy$ such that $Sg \circ u = f$. Hence setting $\hat{f} = g$ make u a universal arrow. □

Corollary 1. Any two universal arrows from x to F can be canonically identified by Lemma 2.

Definition. A *zero object* of $\mathscr{C}$ is an object that is both initial and terminal. A *pointed category* is a category with a chosen zero object.

Example 1. The unique initial object of **Set** is $\emptyset$, and the terminal objects are precisely the singleton sets. Hence there is no zero object. Moreover, there are no initial or terminal objects in iso(**Set**).

Definition. Given $X \in \operatorname{ob} \mathscr{C}$, the *undercategory* $X/\mathscr{C}$ has as objects morphisms in $\mathscr{C}$ of the form $i : X \to Y$ where X is fixed. Given $i : X \to Y$ and $i' : X \to Y'$ in $\operatorname{ob} X/\mathscr{C}$, define the set of morphisms from i to i' as the morphisms $f : Y \to Y'$ where

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ & \searrow^{i'} & \downarrow f \\ & & Y' \end{array}$$

commutes. We call i the *structure morphism.*

Composition and identity carry over exactly from $\mathscr{C}$.

Definition. Given $x \in \operatorname{ob} \mathscr{C}$, The *overcategory* $\mathscr{C}/X$ has as objects morphisms in $\mathscr{C}$ of the form $i : Y \to X$ where X is fixed. Given $i : Y \to X$ and $i' : Y' \to X$ in $\operatorname{ob} \mathscr{C}/X$, define the set of morphisms from i to i' as the morphisms $f : Y \to Y'$ where

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ & \searrow^{i} & \downarrow i' \\ & & X \end{array}$$

commutes. We again call i the *structure morphism.*

Composition and identity carry over exactly from $\mathscr{C}$.

Remark 4. If $X \in \operatorname{ob} \mathscr{C}$, then $(X/\mathscr{C})^{\mathrm{op}} = \mathscr{C}^{\mathrm{op}}/X$. Thus, the under- and overcategories are dual in the sense that $P(X, \mathscr{C})$ is true for any undercategory $X/\mathscr{C}$ iff $P^{\mathrm{op}}(X, \mathscr{C})$ is true for any overcategory $\mathscr{C}/X$.

Lemma 3. For any $X \in \mathscr{C}$, the identity morphism on X is an initial object $X/\mathscr{C}$. Dually, it is a terminal object in $\mathscr{C}/X$.

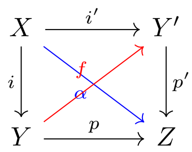
Proof. Any $i : X \to Y$ is itself the unique morphism from Id_X to i. □

Lemma 4. Let X be an initial object of $\mathscr{C}$. The identity morphism on X is a zero object $\mathscr{C}/X$. Dually, if $Y \in \operatorname{ob} \mathscr{C}$ is terminal, then Id_Y is a zero object in $Y/\mathscr{C}$.

Proof. We already know that Id_X is terminal. If $p : Y \to X$ is an object in $\mathscr{C}/X$, then there is a unique morphism $f : X \to Y$. Then $f \circ p$ must equal Id_X. □

Example 2. Let (X, x) be a pointed set with $X = \{x\}$. Let $\mathbf{Set}_*$ denotes the category of pointed sets with base point preserving functions. Then since $\mathbf{Set}_* \cong X/\mathbf{Set}$, it follows that X is a zero object in $\mathbf{Set}_*$.

Definition. Given a morphism $\alpha : X \to Z$ in $\mathscr{C}$, define the *under-and-overcategory* $(X/\mathscr{C}/Z)_\alpha$ as having triples (Y, i, p) as obejcts where $i : X \to Y$ and $p : Y \to Z$ are morphisms in $\mathscr{C}$ such that $p \circ i = \alpha$. Define the set of morphisms from (Y, i, p) to (Y', u', p') as the morphisms $f : Y \to Y'$ such that

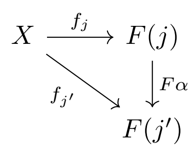


commutes. If $\alpha = \mathrm{Id}_X$, then we call $(X/\mathscr{C}/X)_{\mathrm{Id}_X}$ the category of *retractive* objects over X as each triple (Y, i, p) is a retraction of Y onto X.

Example 3. If $F : \mathscr{C} \to \mathscr{C}$ is the identity functor, then the undercategory $Y/\mathscr{C}$ equals the right fiber category Y/F while the overcategory $\mathscr{C}/Y$ equals the left fiber category F/Y.

Definition. Let $\mathscr{J}$ be a category. A *diagram of shape* $\mathscr{J}$ *in* $\mathscr{C}$ is a functor $F : \mathscr{J} \to \mathscr{C}$.

Definition. Given a functor $F : \mathscr{J} \to \mathscr{C}$ and $X \in \operatorname{ob} \mathscr{C}$, a *cone over* F consists of an *apex* $X \in \operatorname{ob} \mathscr{C}$ and legs $f_j : X \to F(j)$ for each $J \in \operatorname{ob} \mathscr{J}$ such that for any $\alpha : j \to j'$,



commutes. This is just a natural transformation $\Delta_{\mathscr{J}} X \Rightarrow F$ where $\Delta_{\mathscr{J}} X$ denotes the constant functor on $\mathscr{J}$ at X. If $\mathscr{J}$ is small, then $\Delta_{\mathscr{J}}$ is just a functor from $\mathscr{C}$ to $\mathbf{Fun}(\mathscr{J}, \mathscr{C})$.

Definition. The *category of cones over* F is the right fiber category X/F. The *category of cones under* F is the left fiber category F/X.

Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $g : Y \to Z$ a morphism in $\mathscr{D}$. Let $\Delta_{\mathscr{C}} g : \Delta_{\mathscr{C}} Y \Rightarrow \Delta_{\mathscr{C}} Z$ be the natural transformation with components $X \mapsto g$. A *colimit* for the functor $F : \mathscr{C} \to \mathscr{D}$ consists of an object Y of $\mathscr{D}$ and a natural transformation $i : F \Rightarrow \Delta_{\mathscr{C}} Y$ such that for any $Z \in \operatorname{ob} \mathscr{D}$ and natural transformation $j : F \Rightarrow \Delta_{\mathscr{C}} Z$, there is a unique morphism $g : Y \to Z$ such that $j = \Delta_{\mathscr{C}} g \circ i$. We write $Y = \operatorname{colim}_{\mathscr{C}} F$.

Definition. We say that $\mathscr{D}$ admits *all* $\mathscr{C}$*-shaped colimits* if each functor $G : \mathscr{C} \to \mathscr{D}$ has a colimit and that $\mathscr{D}$ is *cocomplete* if each functor $G : \mathscr{C} \to \mathscr{D}$ with $\mathscr{C}$ small has a colimit.

Remark 5. If $\mathscr{C}$ is small, then a colimit of $F : \mathscr{C} \to \mathscr{D}$ is just an initial object in the right fiber category $F/\Delta_{\mathscr{C}}$, which has as objects pairs $(Z, j : F \to \Delta Z)$ and as morphisms from (Y, i) to (Z, j) the morphisms $g : Y \to Z$ in $\mathscr{D}$ such that $\Delta g \circ i = j$.

Remark 6. Notice that there is a natural bijection $\mathscr{D}(Y, Z) \cong \mathbf{Fun}(\mathscr{C}, \mathscr{D})(F, \Delta Z)$ iff $Y = \operatorname{colim}_{\mathscr{C}} F$.

Proposition 2. Any two colimits are canonically isomorphic.

Proof. When $\mathscr{C}$ is small, this is immediate from Lemma 2. But note that the proof of Lemma 2 does not require that $\mathscr{C}$ be locally small (a property which Rognes stipulates of any category). □

Lemma 5. Assume that $\mathscr{D}$ admits all $\mathscr{C}$-shapes colimits and that $\mathscr{C}$ is small. Then a (possibly global) global choice function $\operatorname{colim}_{\mathscr{C}} : \mathbf{Fun}(\mathscr{C}, \mathscr{D}) \to \mathscr{D}$ given by choosing a colimit for each functor determines a functor that is left adjoint to the constant diagram functor $\Delta_{\mathscr{C}} : \mathscr{D} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$.

Proof. For any functor $F : \mathscr{C} \to \mathscr{D}$, there is a bijection $\mathscr{D}(\operatorname{colim}_{\mathscr{C}} F, Z) \cong \mathbf{Fun}(\mathscr{C}, \mathscr{D})(F, \Delta_{\mathscr{C}} Z)$. □

Definition. A limit of the functor $F : \mathscr{C} \to \mathscr{D}$ is the colimit of $F^{op} : \mathscr{C}^{op} \to \mathscr{D}^{op}$.

Remark 7. Explicitly, a limit for $F : \mathscr{C} \to \mathscr{D}$ is an object Z of $\mathscr{D}$ and a natural transformation $p : \Delta_{\mathscr{C}} Z \Rightarrow F$ such that for any $Y \in \operatorname{ob} \mathscr{D}$ and natural transformation $q : \Delta_{\mathscr{C}} Y \Rightarrow F$, there is a unique morphism $g : Y \to Z$ such that $q = p \circ \Delta_{\mathscr{C}} g$.

Remark 8. The colimit of a functor F is the limit of F^{op}. Hence limit and colimit are dual properties, and the above results for colimits can be dualized.

Example 4. If $\mathscr{C}$ is the empty category, then the empty functor $F : \mathscr{C} \to \mathscr{D}$ has $F/\Delta_{\mathscr{C}} \cong \mathscr{D}$, so that the colimit is an initial object of $\mathscr{D}$.

Definition. Let $\mathscr{J}$ be a discrete small category. A diagram of shape $\mathscr{J}$ is a family $\{A_i\}_{i \in J}$. A limit for this diagram is the *product* $\prod_i A_i$ equipped with projections $\pi_i : \prod_i A_i \to A_i$ such that for every $f_i : U \to A_i$ there is some unique $f : U \to \prod_i A_i$ with $\pi_i \circ f = f_i$.

Dually, a colimit for the diagram is the *coproduct* $\sum_i A_i$ equipped with inclusions $u_i : A_i \to \sum_i A_i$ such that for any $f_i : A_i \to Y$, there is some unique $f : \sum_i A_i \to Y$ with $f_i = f \circ u_i$.

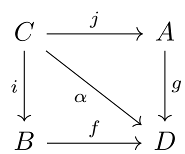
Example 5. Familiar examples include disjoint unions, free products, cartesian products, and direct products.

Definition. Let $\mathscr{J}$ be the category $\bullet \rightrightarrows \bullet$. Then a diagram of shape $\mathscr{J}$ looks like $A \underset{g}{\overset{f}{\rightrightarrows}} B$. A cone over this with apex C and legs $f_1 : C \to A$ and $f_2 : C \to B$ satisfies $ff_1 = f_2 = gf_1$. If such an object C together with f_1 is the limit of the diagram, then we say it is the *equalizer* of f and g. Dually, a colimit is called the *coequalizer*.

Example 6. The equalizer in **Set** of $f, g : X \to Y$ is the subset $X' := \{x \in X : f(x) = g(x)\}$ together with the inclusion $X' \hookrightarrow X$. The coequalizer of (f, g) if $^{Y}\!/_{\sim}$ together with the quotient map on B where $\sim$ is the smallest equivalence relation under which $f(x) \sim g(x)$ for every x.

Example 7. The same idea applies to **Grp**. The relation $\sim$ just becomes a particular minimal normal subgroup.

Definition. Let $\mathscr{J}$ be the category $\bullet \to \bullet \leftarrow \bullet$. Then a diagram of this shape looks like $B \xrightarrow{f} D \xleftarrow{g} A$, while a cone over this diagram looks like



If such an object C together with i and j is the limit of this diagram, then we call it the *pullback* of f and g, denoted by $B \times_D A$.

Definition. We can perform an analogous construction for $\mathscr{J}^{op}$. Then the colimit of the resulting diagram is called the *pushout*, denoted by $B \cup_D A$.

Example 8. The pullback in **Set** of $f : X \to Z$ and $g : Y \to Z$ is the subset $\{(x, y) \in X \times Y : f(x) = g(y)\}$, called the *fiber product* of X and Y over Z.

Theorem 1. (Freyd)

1. If $\mathscr{C}$ has equalizers and all small (resp. finite) products, then it has all small (resp. finite) limits.
2. If $\mathscr{C}$ has pullbacks and a terminal object, then it has all finite limits.

Proof.

1. See Johnstone, Theorem 4.9.

2. By part 1, it suffices to show that $\mathscr{C}$ has equalizers and all finite products. By assumption there is some terminal object 1. Then any product $A_1 \times A_2$ can be realized as the pullback of $A_1 \to 1 \leftarrow A_2$. By induction $\mathscr{C}$ has all finite products. Moreover, for morphisms $f, g : A \to B$, note that any cone over the diagram $A \underset{(1_A, g)}{\longrightarrow} A \times B \underset{(1_A, f)}{\longleftarrow} A$ admits morphisms $h : A \to C$ and $k : C \to A$ such that $h = k$ and $fk = gh$. Thus the pullback for this diagram is an equalizer for (f, g), completing the proof.

□

Corollary 2. Both **Set** and **Grp** are complete and cocomplete (or *bicomplete*).

Remark 9. It turns out that adjoints interact nicely with (co)limits under mild conditions.

Proposition 3. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be an adjoint pair and $\mathscr{E}$ small. If $X : \mathscr{E} \to \mathscr{C}$ is a functor with $\operatorname{colim}_{\mathscr{E}} X$, then

$$\operatorname{colim}_{\mathscr{E}}(F \circ X) = F(\operatorname{colim}_{\mathscr{E}} X).$$

Dually, if $Y : \mathscr{E} \to \mathscr{D}$ is a functor with $\lim_{\mathscr{E}} Y$, then

$$\lim_{\mathscr{E}}(G \circ Y) = G(\lim_{\mathscr{E}} Y).$$

Proof. We have the following chain of natural bijections for each $Y \in \mathscr{D}$:

$$\mathscr{D}(F(\operatorname{colim}_{\mathscr{E}}), Y) \cong \mathscr{C}(\operatorname{colim}_{\mathscr{E}} X, G(Y)) \cong \lim_{\mathscr{E}} \mathscr{C}(X(-), G(Y)) \cong \lim_{\mathscr{E}} \mathscr{D}(F(X(-)), Y) \cong \mathbf{Fun}(\mathscr{E}, \mathscr{D})(F \circ X, \Delta Y).$$

The second bijection follows from the fact that both sets can be identified with the components of the natural transformations from X to $\Delta G(Y)$.

The second claim follows by duality. □

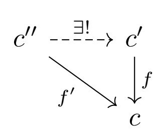
Definition. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. The *fiber category* $F^{-1}(Y)$ is the full subcategory of $\mathscr{C}$ generated by the objects X with $F(X) = Y$.

Definition. Suppose $\mathscr{C}$ has terminal object 1. A *cofiber* of a morphism $f : X \to Y$ is a pushout of the diagram $1 \leftarrow X \to Y$. We write Y/X. Further, given a morphism $p : 1 \to Y$, the *fiber* of f at p is a pullback of $1 \to Y \leftarrow X$. We write $f^{-1}(p)$.

Definition. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. For each $Y \in \operatorname{ob} \mathscr{D}$, there is a full and faithful functor $F^{-1}(Y) \rightarrowtail F/Y$ given by $X \mapsto (X, \mathrm{Id}_Y)$. We say that $\mathscr{C}$ is a *precofibered category* over $\mathscr{D}$ if this functor admits a left adjoint given by $(Z, g : F(Z) \to Y) \mapsto g_*(Z)$.

Moreover, there is a full and faithful functor $F^{-1}(Y) \rightarrowtail Y/F$ defined in the same way. We say that $\mathscr{C}$ is a *prefibered category* over $\mathscr{D}$ if this functor admits a right adjoint given by $(Z, g : Y \to F(Z)) \mapsto g_*(Z)$.

Definition. Let $F; \mathscr{C} \to \mathscr{D}$ be a functor and $f : c' \to c$ be a morphism in $\mathscr{C}$. We say f is *cartesian* if for any morphism $f' : c'' \to c$ in $\mathscr{C}$ and $g : F(c'') \to F(c')$ in $\mathscr{D}$ such that $Ff \circ g = Ff'$, there exists a unique $\phi : c'' \to c$ such that $f' = f \circ \phi$ and $F\phi = g$. In other words, any filling of the following diagram can be lifted to a filling in $\mathscr{D}$.



Definition. We say that F is a *fibration* if for any $c \in \mathscr{C}$ and morphism $f : d \to Fc$, there is a cartesian $\phi : c' \to c$ such that $F\phi = f$. Such ϕ is called a *cartesian lifting* of f to c.

Example 9. Let **Mod** denote the category of left R-modules where R is a ring. Then the forgetful functor $U : \mathbf{Mod} \to \mathbf{Ring}$ is a fibration.